Applied Mathematics Letters 25 (2012) 1267–1269

Contents lists available at SciVerse ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

A note on asymptotic pointwise weaker Meir–Keeler-type contractions

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ARTICLE INFO

Article history:
Received 22 February 2010
Accepted 21 November 2011

Keywords:
Asymptotic pointwise weaker Meir–Keeler-type contraction
Fixed point
Uniformly convex Banach space

ABSTRACT

In this work, we first define the asymptotic pointwise weaker Meir–Keeler-type ψ-contraction, $\psi : X \to \Re^+$, and then a simple proof for the existence of fixed point theorems for the asymptotic pointwise weaker Meir–Keeler-type ψ-contraction is given.

1. Introduction

The notion of asymptotic pointwise mappings was introduced in [1,2]. In 2008, the authors gave simple and elementary proofs for the existence of fixed point theorems for asymptotic pointwise mappings without the use of ultrapowers. In this work, we first define the asymptotic pointwise weaker Meir–Keeler-type ψ-contraction, $\psi : X \to \Re^+$, and then a simple proof for the existence of fixed point theorems for the asymptotic pointwise weaker Meir–Keeler-type ψ-contraction is given.

Throughout this work, by $\Re^+$ we denote the set of all real non-negative numbers, while $\mathbb{N}$ is the set of all natural numbers. Pointwise contractions are defined as follows.

Definition 1. Let (M, d) be a metric space. A mapping $T : M \to M$ is called a pointwise contraction if there exists a mapping $\alpha : M \to [0, 1)$ such that

$$d(Tx, Ty) \le \alpha(x)d(x, y) \quad \text{for each } y \in M.$$

And the fixed point result for the pointwise contraction is the following.

Theorem 1 ([3,4])**.** *Let K be a weakly compact convex subset of a Banach space and suppose $T : K \to K$ is a pointwise contraction. Then T has a unique fixed point $\bar{x}$, and $\{T^n(x)\}$ converges to $\bar{x}$ for each $x \in M$.*

Asymptotic contractions are defined as follows. Let Φ denote the class of all mappings $\phi : \Re^+ \to \Re^+$ satisfying

(i) ϕ is continuous,
(ii) $0 \le \phi(t) < t$ for all $t \in \Re^+ \setminus \{0\}$, $\phi(0) = 0$.

Definition 2. Let (M, d) be a metric space. A mapping $T : M \to M$ is said to be an asymptotic contraction if

$$d(T^n x, T^n y) \le \phi_n(d(x, y)) \quad \text{for all } x, y \in M, \tag{1}$$

where $\phi_n \to \phi \in \Phi$ uniformly on the range of d.

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doi:10.1016/j.aml.2011.11.021

The fixed point result for the asymptotic contraction is the following.

Theorem 2 (*[1]*). *Suppose (M, d) is a complete metric space and suppose $T : M \to M$ is a continuous asymptotic contraction for which the mappings ϕ_n in (1) are continuous. Assume also that some orbit of T is bounded. Then T has a fixed point $z \in M$, and moreover the Picard iterates $\{T^n(x)\}$ converge to z for each $x \in M$.*

Recall the notion of the Meir–Keeler-type function. A function $\psi : \Re^+ \to \Re^+$ is said to be a Meir–Keeler-type function (see [5]) if for each $\eta \in \Re^+$, there exists $\delta > 0$ such that for $t \in \Re^+$ with $\eta \le t < \eta + \delta$, we have $\psi(t) < \eta$. We now define a new notion of the weaker Meir–Keeler-type function, as follows.

Definition 3. The function $\psi : \Re^+ \to \Re^+$ is called a weaker Meir–Keeler-type function if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in \Re^+$ with $\eta \le t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Applying the concept of the weaker Meir–Keeler-type function, we define an asymptotic pointwise weaker Meir–Keeler-type contraction as follows.

Definition 4. Let X be a Banach space, and let $\psi : \Re^+ \to \Re^+$ be a weaker Meir–Keeler-type function. Then the mapping $T : X \to X$ is said to be an asymptotic pointwise weaker Meir–Keeler-type ψ-contraction if for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \le \psi^n(\|x\|)\|x - y\|, \quad \text{for each } x, y \in X.$$

2. The main results

In this study, we also use the technique of asymptotic centers. Let X be a Banach space, A a subset of X, and $\{x_n\}$ a bounded sequence in X. The asymptotic center of $\{x_n\}$ relative to A, denoted as $C_A(x_n)$, is the set of minimizers in A (if any) of the function f given by

$$f(x) = \limsup_{n\to\infty} \|x_n - x\|.$$

That is,

$$C_A(x_n) = \{x \in A : f(x) = inf_A f\}.$$

The technique of asymptotic centers plays an important role in the following main theorem.

Theorem 3. *Let A be a weakly compact convex subset of a Banach space X, let $\psi : \Re^+ \to \Re^+$ be a weaker Meir–Keeler-type function where for each $t \in \Re^+$, $\{\psi^n(t)\}_{n\in\mathbb{N}}$ is nonincreasing, and let $T : A \to A$ be an asymptotic pointwise weaker Meir–Keeler-type ψ-contraction. Then T has a unique fixed point $\bar{x} \in A$, and for each $x \in A$, the sequence of Picard iterates, $\{T^n x\}$, converges in norm to $\bar{x}$.*

Proof. Fix an $x \in A$ and define a function f by

$$f(x) = \limsup_{n\to\infty} \|T^n x - y\|, \quad y \in A.$$

Since A is a weakly compact convex subset of a Banach space X, the asymptotic center of the sequence $\{T^n x\}$ relative to A

$$C_A(T^n x) = \{y \in A : f(y) = min_A f\}$$

is a non-empty closed convex subset of A. We now claim that

$$f(T^m y) \le \psi^m(\|y\|) f(y), \quad y \in A.$$

Indeed, we have

$$\begin{aligned} f(T^m y) &= \limsup_{n\to\infty} \|T^n x - T^m y\| \\ &= \limsup_{n\to\infty} \|T^{m+n} x - T^m y\| \\ &= \limsup_{n\to\infty} \|T^m (T^n x - y)\| \\ &\le \limsup_{n\to\infty} \psi^m(\|y\|)\|T^n x - y\| \\ &= \psi^m(\|y\|) f(y). \end{aligned}$$

Take an $y \in C_A(T^n x)$, and since $T^m y \in A$, we get, for $m \ge 1$,

$$f(y) \le f(T^m y) \le \psi^m(\|y\|) f(y). \tag{2}$$

Since $\{\psi^m(\|y\|)\}_{m\in\mathbb{N}}$ is nonincreasing, it must converge to some $\eta \ge 0$. We claim that $\eta = 0$. To the contrary, assume that $\eta > 0$. Then by the definition of the weaker Meir–Keeler-type function, there exists $\delta > \eta$ such that for $y \in A$ with

$\eta \le \|y\| < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\|y\|) < \eta$. Since $\lim_{m\to\infty} \psi^m(\|y\|) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \psi^m(\|y\|) < \delta$ for all $m \ge m_0$. Thus we conclude that $\psi^{m_0+n_0}(\|y\|) < \eta$, and we get a contradiction. So $\lim_{m\to\infty} \psi^m(\|y\|) = 0$.

Taking the limit in (2) as $m \to \infty$, we get $f(y) = 0$. This together with (2) implies that $f(T^m y) = 0$ for all $m \ge 1$; in particular, $f(Ty) = 0$. Therefore, we have $T^n x \to y$ and $T^n x \to Ty$, both in norm. Hence $Ty = y$. It is easy to see that T has a unique fixed point. Indeed, if $z \in A$ is also a fixed point of T, then for all $n \in \mathbb{N}$,

$$\|y - z\| = \|T^n y - T^n z\| \le \psi^n(\|y\|)\|y - z\|.$$

Letting $n \to \infty$, we get $\|y - z\| = 0$, and so $y = z$. $\square$

Acknowledgment

This research was supported by the National Science Council of the Republic of China.

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